流体力学III追加説明

ラグランジェの渦の不生不滅の法則

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渦関連の法則の解説

- ここでは、授業において部分的に省略しながら解説した「ラグランジェの渦の不生不滅の法則」についての詳細な証明を解説していく。

ラグランジェの運動方程式

証明にはラグランジェの基礎方程式（p.8(1.22)~(1.24)）を使用していく。

$$\frac{\partial x}{\partial x_0}\frac{\partial^2 x}{\partial t^2}+\frac{\partial y}{\partial x_0}\frac{\partial^2 y}{\partial t^2}+\frac{\partial z}{\partial x_0}\frac{\partial^2 z}{\partial t^2}=-\frac{1}{\rho}\frac{\partial p}{\partial x_0}+\frac{\partial x}{\partial x_0}K_x+\frac{\partial y}{\partial x_0}K_y+\frac{\partial z}{\partial x_0}K_z$$

$$\frac{\partial x}{\partial y_0}\frac{\partial^2 x}{\partial t^2}+\frac{\partial y}{\partial y_0}\frac{\partial^2 y}{\partial t^2}+\frac{\partial z}{\partial y_0}\frac{\partial^2 z}{\partial t^2}=-\frac{1}{\rho}\frac{\partial p}{\partial y_0}+\frac{\partial x}{\partial y_0}K_x+\frac{\partial y}{\partial y_0}K_y+\frac{\partial z}{\partial y_0}K_z$$

$$\frac{\partial x}{\partial z_0}\frac{\partial^2 x}{\partial t^2}+\frac{\partial y}{\partial z_0}\frac{\partial^2 y}{\partial t^2}+\frac{\partial z}{\partial z_0}\frac{\partial^2 z}{\partial t^2}=-\frac{1}{\rho}\frac{\partial p}{\partial z_0}+\frac{\partial x}{\partial z_0}K_x+\frac{\partial y}{\partial z_0}K_y+\frac{\partial z}{\partial z_0}K_z$$

仮定

① 外力は保存力であると仮定

$$K_x = -\frac{\partial \Omega}{\partial x} \qquad K_y = -\frac{\partial \Omega}{\partial y} \qquad K_z = -\frac{\partial \Omega}{\partial z}$$

Ωはポテンシャルエネルギー

② バロトロピー流体を仮定

$$\frac{1}{\rho}\frac{\partial p}{\partial x_0} = \frac{\partial P}{\partial x_0} \qquad \frac{1}{\rho}\frac{\partial p}{\partial y_0} = \frac{\partial P}{\partial y_0} \qquad \frac{1}{\rho}\frac{\partial p}{\partial z_0} = \frac{\partial P}{\partial z_0}$$

Pは圧力関数

外力項の変換

外力項をポテンシャルエネルギーにより表現する。

$$\frac{\partial x}{\partial x_0}K_x+\frac{\partial y}{\partial x_0}K_y+\frac{\partial z}{\partial x_0}K_z=-\frac{\partial x}{\partial x_0}\frac{\partial \Omega}{\partial x}-\frac{\partial y}{\partial x_0}\frac{\partial \Omega}{\partial y}-\frac{\partial z}{\partial x_0}\frac{\partial \Omega}{\partial z}=-\frac{\partial \Omega}{\partial x_0}$$

$$\frac{\partial x}{\partial y_0}K_x+\frac{\partial y}{\partial y_0}K_y+\frac{\partial z}{\partial y_0}K_z=-\frac{\partial x}{\partial y_0}\frac{\partial \Omega}{\partial x}-\frac{\partial y}{\partial y_0}\frac{\partial \Omega}{\partial y}-\frac{\partial z}{\partial y_0}\frac{\partial \Omega}{\partial z}=-\frac{\partial \Omega}{\partial y_0}$$

$$\frac{\partial x}{\partial z_0}K_x+\frac{\partial y}{\partial z_0}K_y+\frac{\partial z}{\partial z_0}K_z=-\frac{\partial x}{\partial z_0}\frac{\partial \Omega}{\partial x}-\frac{\partial y}{\partial z_0}\frac{\partial \Omega}{\partial y}-\frac{\partial z}{\partial z_0}\frac{\partial \Omega}{\partial z}=-\frac{\partial \Omega}{\partial z_0}$$

注意点：ラグランジェの運動方程式としてはx, y, z微分は使えず、x_0, y_0, z_0微分の変換する必要がある。

運動方程式の変換

以上の仮定を用いて整理し直すと最終的には以下のようになる。

$$\frac{\partial x}{\partial x_0}\frac{\partial^2 x}{\partial t^2}+\frac{\partial y}{\partial x_0}\frac{\partial^2 y}{\partial t^2}+\frac{\partial z}{\partial x_0}\frac{\partial^2 z}{\partial t^2}=-\frac{\partial(P+\Omega)}{\partial x_0} \quad \text{Eq.1}$$

$$\frac{\partial x}{\partial y_0}\frac{\partial^2 x}{\partial t^2}+\frac{\partial y}{\partial y_0}\frac{\partial^2 y}{\partial t^2}+\frac{\partial z}{\partial y_0}\frac{\partial^2 z}{\partial t^2}=-\frac{\partial(P+\Omega)}{\partial y_0} \quad \text{Eq.2}$$

$$\frac{\partial x}{\partial z_0}\frac{\partial^2 x}{\partial t^2}+\frac{\partial y}{\partial z_0}\frac{\partial^2 y}{\partial t^2}+\frac{\partial z}{\partial z_0}\frac{\partial^2 z}{\partial t^2}=-\frac{\partial(P+\Omega)}{\partial z_0} \quad \text{Eq.3}$$

角運動量保存則

- 速度は運動量の構成要素であり、質量を直接取り扱わない流体力学では代用されている。
- 運動量保存則から運動方程式が導出された。
- 角運動量は渦度に関連していると考えられることから、渦度の方程式を導出する方向で変換していく。

渦度ベクトル定義式

$$\omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

渦度ベクトル成分は速度の空間微分で構成されていることから、運動方程式に速度を導入して変形する。

$$\frac{\partial x}{\partial x_0}\frac{\partial u}{\partial t} + \frac{\partial y}{\partial x_0}\frac{\partial v}{\partial t} + \frac{\partial z}{\partial x_0}\frac{\partial w}{\partial t} = -\frac{\partial(P+\Omega)}{\partial x_0} \qquad \text{Eq.1}$$

$$\frac{\partial x}{\partial y_0}\frac{\partial u}{\partial t} + \frac{\partial y}{\partial y_0}\frac{\partial v}{\partial t} + \frac{\partial z}{\partial y_0}\frac{\partial w}{\partial t} = -\frac{\partial(P+\Omega)}{\partial y_0} \qquad \text{Eq.2}$$

$$\frac{\partial x}{\partial z_0}\frac{\partial u}{\partial t} + \frac{\partial y}{\partial z_0}\frac{\partial v}{\partial t} + \frac{\partial z}{\partial z_0}\frac{\partial w}{\partial t} = -\frac{\partial(P+\Omega)}{\partial z_0} \qquad \text{Eq.3}$$

Eq.1と2の変形.1

Eq.1のy_0微分
$$\frac{\partial}{\partial y_0}\left(\frac{\partial x}{\partial x_0}\frac{\partial u}{\partial t}+\frac{\partial y}{\partial x_0}\frac{\partial v}{\partial t}+\frac{\partial z}{\partial x_0}\frac{\partial w}{\partial t}\right)=-\frac{\partial^2\left(P+\Omega\right)}{\partial x_0\partial y_0}$$

Eq.2のx_0微分
$$\frac{\partial}{\partial x_0}\left(\frac{\partial x}{\partial y_0}\frac{\partial u}{\partial t}+\frac{\partial y}{\partial y_0}\frac{\partial v}{\partial t}+\frac{\partial z}{\partial y_0}\frac{\partial w}{\partial t}\right)=-\frac{\partial^2\left(P+\Omega\right)}{\partial x_0\partial y_0}$$

両式の差

$$\frac{\partial}{\partial y_0}\left(\frac{\partial x}{\partial x_0}\frac{\partial u}{\partial t}+\frac{\partial y}{\partial x_0}\frac{\partial^2 y}{\partial t^2}+\frac{\partial z}{\partial x_0}\frac{\partial^2 z}{\partial t^2}\right)-\frac{\partial}{\partial x_0}\left(\frac{\partial x}{\partial y_0}\frac{\partial u}{\partial t}+\frac{\partial y}{\partial y_0}\frac{\partial^2 y}{\partial t^2}+\frac{\partial z}{\partial y_0}\frac{\partial^2 z}{\partial t^2}\right)=0$$

微分を展開

$$\cancel{\frac{\partial^2 x}{\partial x_0\partial y_0}\frac{\partial u}{\partial t}}+\frac{\partial x}{\partial x_0}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y_0}\right)+\cancel{\frac{\partial^2 y}{\partial x_0\partial y_0}\frac{\partial v}{\partial t}}+\frac{\partial y}{\partial x_0}\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial y_0}\right)+\cancel{\frac{\partial^2 z}{\partial x_0\partial y_0}\frac{\partial w}{\partial t}}+\frac{\partial z}{\partial x_0}\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial y_0}\right)$$

$$-\cancel{\frac{\partial^2 x}{\partial x_0\partial y_0}\frac{\partial u}{\partial t}}-\frac{\partial x}{\partial y_0}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x_0}\right)-\cancel{\frac{\partial^2 y}{\partial x_0\partial y_0}\frac{\partial v}{\partial t}}-\frac{\partial y}{\partial y_0}\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x_0}\right)-\cancel{\frac{\partial^2 z}{\partial x_0\partial y_0}\frac{\partial w}{\partial t}}-\frac{\partial z}{\partial y_0}\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial x_0}\right)=0$$

Eq.1と2の変形.2

$$\frac{\partial x}{\partial x_0}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y_0}\right)+\frac{\partial y}{\partial x_0}\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial y_0}\right)+\frac{\partial z}{\partial x_0}\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial y_0}\right)$$

$$-\frac{\partial x}{\partial y_0}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x_0}\right)-\frac{\partial y}{\partial y_0}\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x_0}\right)-\frac{\partial z}{\partial y_0}\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial x_0}\right)=0$$

時間微分変換

$$\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial x_0}\frac{\partial u}{\partial y_0}\right)-\frac{\partial u}{\partial x_0}\frac{\partial u}{\partial y_0}+\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial x_0}\frac{\partial v}{\partial y_0}\right)-\frac{\partial v}{\partial x_0}\frac{\partial v}{\partial y_0}+\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x_0}\frac{\partial w}{\partial y_0}\right)-\frac{\partial w}{\partial x_0}\frac{\partial w}{\partial y_0}$$

$$-\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial y_0}\frac{\partial u}{\partial x_0}\right)+\frac{\partial u}{\partial y_0}\frac{\partial u}{\partial x_0}-\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial y_0}\frac{\partial v}{\partial x_0}\right)+\frac{\partial v}{\partial y_0}\frac{\partial v}{\partial x_0}-\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial y_0}\frac{\partial w}{\partial x_0}\right)+\frac{\partial w}{\partial y_0}\frac{\partial w}{\partial x_0}=0$$

最終的に

$$\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial x_0}\frac{\partial u}{\partial y_0}-\frac{\partial x}{\partial y_0}\frac{\partial u}{\partial x_0}+\frac{\partial y}{\partial x_0}\frac{\partial v}{\partial y_0}-\frac{\partial y}{\partial y_0}\frac{\partial v}{\partial x_0}+\frac{\partial z}{\partial x_0}\frac{\partial w}{\partial y_0}-\frac{\partial z}{\partial y_0}\frac{\partial w}{\partial x_0}\right)=0$$

Eq.2と3の変形.1

Eq.2のz_0微分
$$\frac{\partial}{\partial z_0}\left(\frac{\partial x}{\partial y_0}\frac{\partial u}{\partial t}+\frac{\partial y}{\partial y_0}\frac{\partial v}{\partial t}+\frac{\partial z}{\partial y_0}\frac{\partial w}{\partial t}\right)=-\frac{\partial^2(P+\Omega)}{\partial y_0\partial z_0}$$

Eq.3のy_0微分
$$\frac{\partial}{\partial y_0}\left(\frac{\partial x}{\partial z_0}\frac{\partial u}{\partial t}+\frac{\partial y}{\partial z_0}\frac{\partial v}{\partial t}+\frac{\partial z}{\partial z_0}\frac{\partial w}{\partial t}\right)=-\frac{\partial^2(P+\Omega)}{\partial y_0\partial z_0}$$

両式の差

$$\frac{\partial}{\partial z_0}\left(\frac{\partial x}{\partial y_0}\frac{\partial u}{\partial t}+\frac{\partial y}{\partial y_0}\frac{\partial v}{\partial t}+\frac{\partial z}{\partial y_0}\frac{\partial w}{\partial t}\right)-\frac{\partial}{\partial y_0}\left(\frac{\partial x}{\partial z_0}\frac{\partial u}{\partial t}+\frac{\partial y}{\partial z_0}\frac{\partial v}{\partial t}+\frac{\partial z}{\partial z_0}\frac{\partial w}{\partial t}\right)=0$$

微分を展開

$$\frac{\partial^2 x}{\partial y_0\partial z_0}\frac{\partial u}{\partial t}+\frac{\partial x}{\partial y_0}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z_0}\right)+\frac{\partial^2 y}{\partial y_0\partial z_0}\frac{\partial v}{\partial t}+\frac{\partial y}{\partial y_0}\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial z_0}\right)+\frac{\partial^2 z}{\partial y_0\partial z_0}\frac{\partial w}{\partial t}+\frac{\partial z}{\partial y_0}\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial z_0}\right)$$
$$-\frac{\partial^2 x}{\partial y_0\partial z_0}\frac{\partial u}{\partial t}-\frac{\partial x}{\partial z_0}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y_0}\right)-\frac{\partial^2 y}{\partial y_0\partial z_0}\frac{\partial v}{\partial t}-\frac{\partial y}{\partial z_0}\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial y_0}\right)-\frac{\partial^2 z}{\partial y_0\partial z_0}\frac{\partial w}{\partial t}-\frac{\partial z}{\partial z_0}\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial y_0}\right)=0$$

Eq.2と3の変形.2

$$\frac{\partial x}{\partial y_0}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z_0}\right)+\frac{\partial y}{\partial y_0}\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial z_0}\right)+\frac{\partial z}{\partial y_0}\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial z_0}\right)$$

$$-\frac{\partial x}{\partial z_0}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y_0}\right)-\frac{\partial y}{\partial z_0}\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial y_0}\right)-\frac{\partial z}{\partial z_0}\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial y_0}\right)=0$$

時間微分変換

$$\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial y_0}\frac{\partial u}{\partial z_0}\right)-\frac{\partial u}{\partial y_0}\frac{\partial u}{\partial z_0}+\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial y_0}\frac{\partial v}{\partial z_0}\right)-\frac{\partial v}{\partial y_0}\frac{\partial v}{\partial z_0}+\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial y_0}\frac{\partial w}{\partial z_0}\right)-\frac{\partial w}{\partial y_0}\frac{\partial w}{\partial z_0}$$

$$-\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial z_0}\frac{\partial u}{\partial y_0}\right)+\frac{\partial u}{\partial z_0}\frac{\partial u}{\partial y_0}-\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial z_0}\frac{\partial v}{\partial y_0}\right)+\frac{\partial v}{\partial z_0}\frac{\partial v}{\partial y_0}-\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial z_0}\frac{\partial w}{\partial y_0}\right)+\frac{\partial w}{\partial z_0}\frac{\partial w}{\partial y_0}=0$$

最終的に

$$\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial y_0}\frac{\partial u}{\partial z_0}-\frac{\partial x}{\partial z_0}\frac{\partial u}{\partial y_0}+\frac{\partial y}{\partial y_0}\frac{\partial v}{\partial z_0}-\frac{\partial y}{\partial z_0}\frac{\partial v}{\partial y_0}+\frac{\partial z}{\partial y_0}\frac{\partial w}{\partial z_0}-\frac{\partial z}{\partial z_0}\frac{\partial w}{\partial y_0}\right)=0$$

Eq.3と1の変形.1

Eq.3のx_0微分
$$\frac{\partial}{\partial x_0}\left(\frac{\partial x}{\partial z_0}\frac{\partial u}{\partial t}+\frac{\partial y}{\partial z_0}\frac{\partial v}{\partial t}+\frac{\partial z}{\partial z_0}\frac{\partial w}{\partial t}\right)=-\frac{\partial^2(P+\Omega)}{\partial z_0\partial x_0}$$

Eq.1のz_0微分
$$\frac{\partial}{\partial z_0}\left(\frac{\partial x}{\partial x_0}\frac{\partial u}{\partial t}+\frac{\partial y}{\partial x_0}\frac{\partial v}{\partial t}+\frac{\partial z}{\partial x_0}\frac{\partial w}{\partial t}\right)=-\frac{\partial^2(P+\Omega)}{\partial z_0\partial x_0}$$

両式の差

$$\frac{\partial}{\partial x_0}\left(\frac{\partial x}{\partial z_0}\frac{\partial u}{\partial t}+\frac{\partial y}{\partial z_0}\frac{\partial v}{\partial t}+\frac{\partial z}{\partial z_0}\frac{\partial w}{\partial t}\right)-\frac{\partial}{\partial z_0}\left(\frac{\partial x}{\partial x_0}\frac{\partial u}{\partial t}+\frac{\partial y}{\partial x_0}\frac{\partial v}{\partial t}+\frac{\partial z}{\partial x_0}\frac{\partial w}{\partial t}\right)=0$$

微分を展開

$$\frac{\partial^2 x}{\partial z_0\partial x_0}\frac{\partial u}{\partial t}+\frac{\partial x}{\partial z_0}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x_0}\right)+\frac{\partial^2 y}{\partial z_0\partial x_0}\frac{\partial v}{\partial t}+\frac{\partial y}{\partial z_0}\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x_0}\right)+\frac{\partial^2 z}{\partial z_0\partial x_0}\frac{\partial w}{\partial t}+\frac{\partial z}{\partial z_0}\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial x_0}\right)$$
$$-\frac{\partial^2 x}{\partial z_0\partial x_0}\frac{\partial u}{\partial t}-\frac{\partial x}{\partial x_0}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z_0}\right)-\frac{\partial^2 y}{\partial z_0\partial x_0}\frac{\partial v}{\partial t}-\frac{\partial y}{\partial x_0}\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial z_0}\right)-\frac{\partial^2 z}{\partial z_0\partial x_0}\frac{\partial w}{\partial t}-\frac{\partial z}{\partial x_0}\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial z_0}\right)=0$$

Eq.3と1の変形.2

$$\frac{\partial x}{\partial z_0}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x_0}\right)+\frac{\partial y}{\partial z_0}\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x_0}\right)+\frac{\partial z}{\partial z_0}\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial x_0}\right)$$

$$-\frac{\partial x}{\partial x_0}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z_0}\right)-\frac{\partial y}{\partial x_0}\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial z_0}\right)-\frac{\partial z}{\partial x_0}\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial z_0}\right)=0$$

時間微分変換

$$\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial z_0}\frac{\partial u}{\partial x_0}\right)-\cancel{\frac{\partial u}{\partial z_0}\frac{\partial u}{\partial x_0}}+\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial z_0}\frac{\partial v}{\partial x_0}\right)-\cancel{\frac{\partial v}{\partial z_0}\frac{\partial v}{\partial x_0}}+\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial z_0}\frac{\partial w}{\partial x_0}\right)-\cancel{\frac{\partial w}{\partial z_0}\frac{\partial w}{\partial x_0}}$$

$$-\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial x_0}\frac{\partial u}{\partial z_0}\right)+\cancel{\frac{\partial u}{\partial x_0}\frac{\partial u}{\partial z_0}}-\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial x_0}\frac{\partial v}{\partial z_0}\right)+\cancel{\frac{\partial v}{\partial x_0}\frac{\partial v}{\partial z_0}}-\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x_0}\frac{\partial w}{\partial z_0}\right)+\cancel{\frac{\partial w}{\partial x_0}\frac{\partial w}{\partial z_0}}=0$$

最終的に

$$\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial z_0}\frac{\partial u}{\partial x_0}-\frac{\partial x}{\partial x_0}\frac{\partial u}{\partial z_0}+\frac{\partial y}{\partial z_0}\frac{\partial v}{\partial x_0}-\frac{\partial y}{\partial x_0}\frac{\partial v}{\partial z_0}+\frac{\partial z}{\partial z_0}\frac{\partial w}{\partial x_0}-\frac{\partial z}{\partial x_0}\frac{\partial w}{\partial z_0}\right)=0$$

変形された運動方程式

以上の変形により、運動方程式は時間微分で括って次式のようにまとめられる。

$$\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial x_0}\frac{\partial u}{\partial y_0}-\frac{\partial x}{\partial y_0}\frac{\partial u}{\partial x_0}+\frac{\partial y}{\partial x_0}\frac{\partial v}{\partial y_0}-\frac{\partial y}{\partial y_0}\frac{\partial v}{\partial x_0}+\frac{\partial z}{\partial x_0}\frac{\partial w}{\partial y_0}-\frac{\partial z}{\partial y_0}\frac{\partial w}{\partial x_0}\right)=0 \qquad \text{Eq.4}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial y_0}\frac{\partial u}{\partial z_0}-\frac{\partial x}{\partial z_0}\frac{\partial u}{\partial y_0}+\frac{\partial y}{\partial y_0}\frac{\partial v}{\partial z_0}-\frac{\partial y}{\partial z_0}\frac{\partial v}{\partial y_0}+\frac{\partial z}{\partial y_0}\frac{\partial w}{\partial z_0}-\frac{\partial z}{\partial z_0}\frac{\partial w}{\partial y_0}\right)=0 \qquad \text{Eq.5}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial z_0}\frac{\partial u}{\partial x_0}-\frac{\partial x}{\partial x_0}\frac{\partial u}{\partial z_0}+\frac{\partial y}{\partial z_0}\frac{\partial v}{\partial x_0}-\frac{\partial y}{\partial x_0}\frac{\partial v}{\partial z_0}+\frac{\partial z}{\partial z_0}\frac{\partial w}{\partial x_0}-\frac{\partial z}{\partial x_0}\frac{\partial w}{\partial z_0}\right)=0 \qquad \text{Eq.6}$$

注意点：渦度ベクトルを導入するためには、一時的にx, y, zによる空間微分を復活させる必要がある。

速度空間微分の変換式

$$\frac{\partial u}{\partial x_0}=\frac{\partial x}{\partial x_0}\frac{\partial u}{\partial x}+\frac{\partial y}{\partial x_0}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial x_0}\frac{\partial u}{\partial z}$$

$$\frac{\partial u}{\partial y_0}=\frac{\partial x}{\partial y_0}\frac{\partial u}{\partial x}+\frac{\partial y}{\partial y_0}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial y_0}\frac{\partial u}{\partial z}$$

$$\frac{\partial u}{\partial z_0}=\frac{\partial x}{\partial z_0}\frac{\partial u}{\partial x}+\frac{\partial y}{\partial z_0}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial z_0}\frac{\partial u}{\partial z}$$

$$\frac{\partial v}{\partial x_0}=\frac{\partial x}{\partial x_0}\frac{\partial v}{\partial x}+\frac{\partial y}{\partial x_0}\frac{\partial v}{\partial y}+\frac{\partial z}{\partial x_0}\frac{\partial v}{\partial z}$$

$$\frac{\partial v}{\partial y_0}=\frac{\partial x}{\partial y_0}\frac{\partial v}{\partial x}+\frac{\partial y}{\partial y_0}\frac{\partial v}{\partial y}+\frac{\partial z}{\partial y_0}\frac{\partial v}{\partial z}$$

$$\frac{\partial v}{\partial z_0}=\frac{\partial x}{\partial z_0}\frac{\partial v}{\partial x}+\frac{\partial y}{\partial z_0}\frac{\partial v}{\partial y}+\frac{\partial z}{\partial z_0}\frac{\partial v}{\partial z}$$

$$\frac{\partial w}{\partial x_0}=\frac{\partial x}{\partial x_0}\frac{\partial w}{\partial x}+\frac{\partial y}{\partial x_0}\frac{\partial w}{\partial y}+\frac{\partial z}{\partial x_0}\frac{\partial w}{\partial z}$$

$$\frac{\partial w}{\partial y_0}=\frac{\partial x}{\partial y_0}\frac{\partial w}{\partial x}+\frac{\partial y}{\partial y_0}\frac{\partial w}{\partial y}+\frac{\partial z}{\partial y_0}\frac{\partial w}{\partial z}$$

$$\frac{\partial w}{\partial z_0}=\frac{\partial x}{\partial z_0}\frac{\partial w}{\partial x}+\frac{\partial y}{\partial z_0}\frac{\partial w}{\partial y}+\frac{\partial z}{\partial z_0}\frac{\partial w}{\partial z}$$

注意点：、x_0, y_0, z_0微分をx, y, z微分に変換する。変換する必要がある。

Eq.4の変形.1

Eq.4における空間微分を変換

$$
\begin{aligned}
\frac{\partial}{\partial t}\Bigg(&\frac{\partial x}{\partial x_0}\frac{\partial x}{\partial y_0}\frac{\partial u}{\partial x}+\frac{\partial x}{\partial x_0}\frac{\partial y}{\partial y_0}\frac{\partial u}{\partial y}+\frac{\partial x}{\partial x_0}\frac{\partial z}{\partial y_0}\frac{\partial u}{\partial z}-\frac{\partial x}{\partial y_0}\frac{\partial x}{\partial x_0}\frac{\partial u}{\partial x}-\frac{\partial x}{\partial y_0}\frac{\partial y}{\partial x_0}\frac{\partial u}{\partial y}\\
&-\frac{\partial x}{\partial y_0}\frac{\partial z}{\partial x_0}\frac{\partial u}{\partial z}+\frac{\partial y}{\partial x_0}\frac{\partial x}{\partial y_0}\frac{\partial v}{\partial x}+\frac{\partial y}{\partial x_0}\frac{\partial y}{\partial y_0}\frac{\partial v}{\partial y}+\frac{\partial y}{\partial x_0}\frac{\partial z}{\partial y_0}\frac{\partial v}{\partial z}-\frac{\partial y}{\partial y_0}\frac{\partial x}{\partial x_0}\frac{\partial v}{\partial x}\\
&-\frac{\partial y}{\partial y_0}\frac{\partial y}{\partial x_0}\frac{\partial v}{\partial y}-\frac{\partial y}{\partial y_0}\frac{\partial z}{\partial x_0}\frac{\partial v}{\partial z}+\frac{\partial z}{\partial x_0}\frac{\partial x}{\partial y_0}\frac{\partial w}{\partial x}+\frac{\partial z}{\partial x_0}\frac{\partial y}{\partial y_0}\frac{\partial w}{\partial y}+\frac{\partial z}{\partial x_0}\frac{\partial z}{\partial y_0}\frac{\partial w}{\partial z}\\
&-\frac{\partial z}{\partial y_0}\frac{\partial x}{\partial x_0}\frac{\partial w}{\partial x}-\frac{\partial z}{\partial y_0}\frac{\partial y}{\partial x_0}\frac{\partial w}{\partial y}-\frac{\partial z}{\partial y_0}\frac{\partial z}{\partial x_0}\frac{\partial w}{\partial z}\Bigg)=0
\end{aligned}
$$

消去後共通因子によって整理

Eq.4の変形.2

$$\frac{\partial}{\partial t}\left\{\left(\frac{\partial z}{\partial x_0}\frac{\partial y}{\partial y_0}-\frac{\partial z}{\partial y_0}\frac{\partial y}{\partial x_0}\right)\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)+\left(\frac{\partial x}{\partial x_0}\frac{\partial z}{\partial y_0}-\frac{\partial x}{\partial y_0}\frac{\partial z}{\partial x_0}\right)\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)\right.$$

$$\left.+\left(\frac{\partial y}{\partial x_0}\frac{\partial x}{\partial y_0}-\frac{\partial y}{\partial y_0}\frac{\partial x}{\partial x_0}\right)\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right\}=0$$

最終的に

$$\frac{\partial}{\partial t}\left(\frac{\partial(z,y)}{\partial(x_0,y_0)}\omega_x+\frac{\partial(x,z)}{\partial(x_0,y_0)}\omega_y+\frac{\partial(y,x)}{\partial(x_0,y_0)}\omega_z\right)=0$$

Eq.5の変形. 1

Eq.5における空間微分を変換

$$
\begin{aligned}
\frac{\partial}{\partial t}\Bigg(&\frac{\partial x}{\partial y_0}\frac{\partial x}{\partial z_0}\frac{\partial u}{\partial x}+\frac{\partial x}{\partial y_0}\frac{\partial y}{\partial z_0}\frac{\partial u}{\partial y}+\frac{\partial x}{\partial y_0}\frac{\partial z}{\partial z_0}\frac{\partial u}{\partial z}-\frac{\partial x}{\partial z_0}\frac{\partial x}{\partial y_0}\frac{\partial u}{\partial x}-\frac{\partial x}{\partial z_0}\frac{\partial y}{\partial y_0}\frac{\partial u}{\partial y}\\
&-\frac{\partial x}{\partial z_0}\frac{\partial z}{\partial y_0}\frac{\partial u}{\partial z}+\frac{\partial y}{\partial y_0}\frac{\partial x}{\partial z_0}\frac{\partial v}{\partial x}+\frac{\partial y}{\partial y_0}\frac{\partial y}{\partial z_0}\frac{\partial v}{\partial y}+\frac{\partial y}{\partial y_0}\frac{\partial z}{\partial z_0}\frac{\partial v}{\partial z}-\frac{\partial y}{\partial z_0}\frac{\partial x}{\partial y_0}\frac{\partial v}{\partial x}\\
&-\frac{\partial y}{\partial z_0}\frac{\partial y}{\partial y_0}\frac{\partial v}{\partial y}-\frac{\partial y}{\partial z_0}\frac{\partial z}{\partial y_0}\frac{\partial v}{\partial z}+\frac{\partial z}{\partial y_0}\frac{\partial x}{\partial z_0}\frac{\partial w}{\partial x}+\frac{\partial z}{\partial y_0}\frac{\partial y}{\partial z_0}\frac{\partial w}{\partial y}+\frac{\partial z}{\partial y_0}\frac{\partial z}{\partial z_0}\frac{\partial w}{\partial z}\\
&-\frac{\partial z}{\partial z_0}\frac{\partial x}{\partial y_0}\frac{\partial w}{\partial x}-\frac{\partial z}{\partial z_0}\frac{\partial y}{\partial y_0}\frac{\partial w}{\partial y}-\frac{\partial z}{\partial z_0}\frac{\partial z}{\partial y_0}\frac{\partial w}{\partial z}\Bigg)=0
\end{aligned}
$$

消去後共通因子によって整理

Eq.5の変形.2

$$\frac{\partial}{\partial t}\left\{\left(\frac{\partial z}{\partial y_0}\frac{\partial y}{\partial z_0}-\frac{\partial z}{\partial z_0}\frac{\partial y}{\partial y_0}\right)\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)+\left(\frac{\partial x}{\partial y_0}\frac{\partial z}{\partial z_0}-\frac{\partial x}{\partial z_0}\frac{\partial z}{\partial y_0}\right)\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)\right.$$

$$\left.+\left(\frac{\partial y}{\partial y_0}\frac{\partial x}{\partial z_0}-\frac{\partial y}{\partial z_0}\frac{\partial x}{\partial y_0}\right)\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right\}=0$$

最終的に

$$\frac{\partial}{\partial t}\left(\frac{\partial(z,y)}{\partial(y_0,z_0)}\omega_x+\frac{\partial(x,z)}{\partial(y_0,z_0)}\omega_y+\frac{\partial(y,x)}{\partial(y_0,z_0)}\omega_z\right)=0$$

Eq.6の変形.1

Eq.6における空間微分を変換

$$
\begin{aligned}
\frac{\partial}{\partial t}\Bigg(&\frac{\partial x}{\partial z_0}\frac{\partial x}{\partial x_0}\frac{\partial u}{\partial x}+\frac{\partial x}{\partial z_0}\frac{\partial y}{\partial x_0}\frac{\partial u}{\partial y}+\frac{\partial x}{\partial z_0}\frac{\partial z}{\partial x_0}\frac{\partial u}{\partial z}-\frac{\partial x}{\partial x_0}\frac{\partial x}{\partial z_0}\frac{\partial u}{\partial x}-\frac{\partial x}{\partial x_0}\frac{\partial y}{\partial z_0}\frac{\partial u}{\partial y}\\
&-\frac{\partial x}{\partial x_0}\frac{\partial z}{\partial z_0}\frac{\partial u}{\partial z}+\frac{\partial y}{\partial z_0}\frac{\partial x}{\partial x_0}\frac{\partial v}{\partial x}+\frac{\partial y}{\partial z_0}\frac{\partial y}{\partial x_0}\frac{\partial v}{\partial y}+\frac{\partial y}{\partial z_0}\frac{\partial z}{\partial x_0}\frac{\partial v}{\partial z}-\frac{\partial y}{\partial x_0}\frac{\partial x}{\partial z_0}\frac{\partial v}{\partial x}\\
&-\frac{\partial y}{\partial x_0}\frac{\partial y}{\partial z_0}\frac{\partial v}{\partial y}-\frac{\partial y}{\partial x_0}\frac{\partial z}{\partial z_0}\frac{\partial v}{\partial z}+\frac{\partial z}{\partial z_0}\frac{\partial x}{\partial x_0}\frac{\partial w}{\partial x}+\frac{\partial z}{\partial z_0}\frac{\partial y}{\partial x_0}\frac{\partial w}{\partial y}+\frac{\partial z}{\partial z_0}\frac{\partial z}{\partial x_0}\frac{\partial w}{\partial z}\\
&-\frac{\partial z}{\partial x_0}\frac{\partial x}{\partial z_0}\frac{\partial w}{\partial x}-\frac{\partial z}{\partial x_0}\frac{\partial y}{\partial z_0}\frac{\partial w}{\partial y}-\frac{\partial z}{\partial x_0}\frac{\partial z}{\partial z_0}\frac{\partial w}{\partial z}\Bigg)=0
\end{aligned}
$$

消去後共通因子によって整理

Eq.6の変形.2

$$\frac{\partial}{\partial t}\left\{\left(\frac{\partial z}{\partial z_0}\frac{\partial y}{\partial x_0}-\frac{\partial z}{\partial x_0}\frac{\partial y}{\partial z_0}\right)\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)+\left(\frac{\partial x}{\partial z_0}\frac{\partial z}{\partial x_0}-\frac{\partial x}{\partial x_0}\frac{\partial z}{\partial z_0}\right)\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)\right.$$

$$\left.+\left(\frac{\partial y}{\partial z_0}\frac{\partial x}{\partial x_0}-\frac{\partial y}{\partial x_0}\frac{\partial x}{\partial z_0}\right)\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right\}=0$$

最終的に

$$\frac{\partial}{\partial t}\left(\frac{\partial(z,y)}{\partial(z_0,x_0)}\omega_x+\frac{\partial(x,z)}{\partial(z_0,x_0)}\omega_y+\frac{\partial(y,x)}{\partial(z_0,x_0)}\omega_z\right)=0$$

Eqs.4～6の積分

$$\frac{\partial}{\partial t}\left(\frac{\partial(z,y)}{\partial(x_0,y_0)}\omega_x+\frac{\partial(x,z)}{\partial(x_0,y_0)}\omega_y+\frac{\partial(y,x)}{\partial(x_0,y_0)}\omega_z\right)=0$$

$$\frac{\partial}{\partial t}\left(\frac{\partial(z,y)}{\partial(y_0,z_0)}\omega_x+\frac{\partial(x,z)}{\partial(y_0,z_0)}\omega_y+\frac{\partial(y,x)}{\partial(y_0,z_0)}\omega_z\right)=0$$

$$\frac{\partial}{\partial t}\left(\frac{\partial(z,y)}{\partial(z_0,x_0)}\omega_x+\frac{\partial(x,z)}{\partial(z_0,x_0)}\omega_y+\frac{\partial(y,x)}{\partial(z_0,x_0)}\omega_z\right)=0$$

時間積分 ⇒

$$\frac{\partial(z,y)}{\partial(x_0,y_0)}\omega_x+\frac{\partial(x,z)}{\partial(x_0,y_0)}\omega_y+\frac{\partial(y,x)}{\partial(x_0,y_0)}\omega_z=const$$

$$\frac{\partial(z,y)}{\partial(y_0,z_0)}\omega_x+\frac{\partial(x,z)}{\partial(y_0,z_0)}\omega_y+\frac{\partial(y,x)}{\partial(y_0,z_0)}\omega_z=const$$

$$\frac{\partial(z,y)}{\partial(z_0,x_0)}\omega_x+\frac{\partial(x,z)}{\partial(z_0,x_0)}\omega_y+\frac{\partial(y,x)}{\partial(z_0,x_0)}\omega_z=const$$

不定積分定数の決定

$t=t_0$

$$const = \frac{\partial(z_0,y_0)}{\partial(x_0,y_0)}\omega_{x0} + \frac{\partial(x_0,z_0)}{\partial(x_0,y_0)}\omega_{y0} + \frac{\partial(y_0,x_0)}{\partial(x_0,y_0)}\omega_{z0} = -\omega_{z0}$$

$$const = \frac{\partial(z_0,y_0)}{\partial(y_0,z_0)}\omega_{x0} + \frac{\partial(x_0,z_0)}{\partial(y_0,z_0)}\omega_{y0} + \frac{\partial(y_0,x_0)}{\partial(y_0,z_0)}\omega_{z0} = -\omega_{x0}$$

$$const = \frac{\partial(z_0,y_0)}{\partial(z_0,x_0)}\omega_{x0} + \frac{\partial(x_0,z_0)}{\partial(z_0,x_0)}\omega_{y0} + \frac{\partial(y_0,x_0)}{\partial(z_0,x_0)}\omega_{z0} = -\omega_{y0}$$

最終的に

$$\frac{\partial(y,z)}{\partial(x_0,y_0)}\omega_x + \frac{\partial(z,x)}{\partial(x_0,y_0)}\omega_y + \frac{\partial(x,y)}{\partial(x_0,y_0)}\omega_z = \omega_{x0}$$

$$\frac{\partial(y,z)}{\partial(y_0,z_0)}\omega_x + \frac{\partial(z,x)}{\partial(y_0,z_0)}\omega_y + \frac{\partial(x,y)}{\partial(y_0,z_0)}\omega_z = \omega_{y0}$$

$$\frac{\partial(y,z)}{\partial(z_0,x_0)}\omega_x + \frac{\partial(z,x)}{\partial(z_0,x_0)}\omega_y + \frac{\partial(x,y)}{\partial(z_0,x_0)}\omega_z = \omega_{z0}$$

注意点：分子を入れ替えてある。

行列表現

$$
\begin{pmatrix} \omega_{x0} \\ \omega_{y0} \\ \omega_{z0} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial(y,z)}{\partial(x_0,y_0)} & \dfrac{\partial(z,x)}{\partial(x_0,y_0)} & \dfrac{\partial(x,y)}{\partial(x_0,y_0)} \\ \dfrac{\partial(y,z)}{\partial(y_0,z_0)} & \dfrac{\partial(z,x)}{\partial(y_0,z_0)} & \dfrac{\partial(x,y)}{\partial(y_0,z_0)} \\ \dfrac{\partial(y,z)}{\partial(z_0,x_0)} & \dfrac{\partial(z,x)}{\partial(z_0,x_0)} & \dfrac{\partial(x,y)}{\partial(z_0,x_0)} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}
$$

$$= \mathbf{A}$$

この表現を時間経過に沿うように逆に解く必要がある。

３行３列の逆行列の公式

$$\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\mathbf{A}^{-1}=\frac{1}{\det\mathbf{A}}\begin{pmatrix} a_{22}a_{33}-a_{23}a_{32} & a_{13}a_{32}-a_{12}a_{33} & a_{12}a_{23}-a_{13}a_{22} \\ a_{23}a_{31}-a_{21}a_{33} & a_{11}a_{33}-a_{13}a_{31} & a_{13}a_{21}-a_{11}a_{23} \\ a_{21}a_{32}-a_{22}a_{31} & a_{12}a_{31}-a_{11}a_{32} & a_{11}a_{22}-a_{12}a_{21} \end{pmatrix}$$

$$\det\mathbf{A}=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}-a_{12}a_{21}a_{33}-a_{11}a_{23}a_{32}$$

注意点：線形代数の公式なので覚えている人には提示不要であるが、一応念のため記載している。

行列式

$$
\begin{aligned}
\det \mathbf{A} = & \left(\frac{\partial y}{\partial y_0}\frac{\partial z}{\partial z_0}-\frac{\partial y}{\partial z_0}\frac{\partial z}{\partial y_0}\right)\left(\frac{\partial z}{\partial z_0}\frac{\partial x}{\partial x_0}-\frac{\partial z}{\partial x_0}\frac{\partial x}{\partial z_0}\right)\left(\frac{\partial x}{\partial x_0}\frac{\partial y}{\partial y_0}-\frac{\partial x}{\partial y_0}\frac{\partial y}{\partial x_0}\right) \\
& +\left(\frac{\partial z}{\partial y_0}\frac{\partial x}{\partial z_0}-\frac{\partial z}{\partial z_0}\frac{\partial x}{\partial y_0}\right)\left(\frac{\partial x}{\partial z_0}\frac{\partial y}{\partial x_0}-\frac{\partial x}{\partial x_0}\frac{\partial y}{\partial z_0}\right)\left(\frac{\partial y}{\partial x_0}\frac{\partial z}{\partial y_0}-\frac{\partial y}{\partial y_0}\frac{\partial z}{\partial x_0}\right) \\
& +\left(\frac{\partial x}{\partial y_0}\frac{\partial y}{\partial z_0}-\frac{\partial x}{\partial z_0}\frac{\partial y}{\partial y_0}\right)\left(\frac{\partial y}{\partial z_0}\frac{\partial z}{\partial x_0}-\frac{\partial y}{\partial x_0}\frac{\partial z}{\partial z_0}\right)\left(\frac{\partial z}{\partial x_0}\frac{\partial x}{\partial y_0}-\frac{\partial z}{\partial y_0}\frac{\partial x}{\partial x_0}\right) \\
& -\left(\frac{\partial x}{\partial y_0}\frac{\partial y}{\partial z_0}-\frac{\partial x}{\partial z_0}\frac{\partial y}{\partial y_0}\right)\left(\frac{\partial z}{\partial z_0}\frac{\partial x}{\partial x_0}-\frac{\partial z}{\partial x_0}\frac{\partial x}{\partial z_0}\right)\left(\frac{\partial y}{\partial x_0}\frac{\partial z}{\partial y_0}-\frac{\partial y}{\partial y_0}\frac{\partial z}{\partial x_0}\right) \\
& -\left(\frac{\partial z}{\partial y_0}\frac{\partial x}{\partial z_0}-\frac{\partial z}{\partial z_0}\frac{\partial x}{\partial y_0}\right)\left(\frac{\partial y}{\partial z_0}\frac{\partial z}{\partial x_0}-\frac{\partial y}{\partial x_0}\frac{\partial z}{\partial z_0}\right)\left(\frac{\partial x}{\partial x_0}\frac{\partial y}{\partial y_0}-\frac{\partial x}{\partial y_0}\frac{\partial y}{\partial x_0}\right) \\
& -\left(\frac{\partial y}{\partial y_0}\frac{\partial z}{\partial z_0}-\frac{\partial y}{\partial z_0}\frac{\partial z}{\partial y_0}\right)\left(\frac{\partial x}{\partial z_0}\frac{\partial y}{\partial x_0}-\frac{\partial x}{\partial x_0}\frac{\partial y}{\partial z_0}\right)\left(\frac{\partial z}{\partial x_0}\frac{\partial x}{\partial y_0}-\frac{\partial z}{\partial y_0}\frac{\partial x}{\partial x_0}\right)
\end{aligned}
$$

行列式の因数分解と逆行列成分.1

$$\det \mathbf{A} = \left(\frac{\partial x}{\partial x_0}\frac{\partial y}{\partial y_0}\frac{\partial z}{\partial z_0} + \frac{\partial x}{\partial y_0}\frac{\partial y}{\partial z_0}\frac{\partial z}{\partial x_0} + \frac{\partial x}{\partial z_0}\frac{\partial y}{\partial x_0}\frac{\partial z}{\partial y_0} \right.$$

$$\left. - \frac{\partial x}{\partial x_0}\frac{\partial y}{\partial z_0}\frac{\partial z}{\partial y_0} - \frac{\partial x}{\partial y_0}\frac{\partial y}{\partial x_0}\frac{\partial z}{\partial z_0} - \frac{\partial x}{\partial z_0}\frac{\partial y}{\partial y_0}\frac{\partial z}{\partial x_0} \right)^2 = {\frac{\partial(x,y,z)}{\partial(x_0,y_0,z_0)}}^{2}$$

11成分

$$a_{22}a_{33} - a_{23}a_{32} = \left(\frac{\partial z}{\partial z_0}\frac{\partial x}{\partial x_0} - \frac{\partial z}{\partial x_0}\frac{\partial x}{\partial z_0} \right)\left(\frac{\partial x}{\partial x_0}\frac{\partial y}{\partial y_0} - \frac{\partial x}{\partial y_0}\frac{\partial y}{\partial x_0} \right)$$

$$- \left(\frac{\partial x}{\partial z_0}\frac{\partial y}{\partial x_0} - \frac{\partial x}{\partial x_0}\frac{\partial y}{\partial z_0} \right)\left(\frac{\partial z}{\partial x_0}\frac{\partial x}{\partial y_0} - \frac{\partial z}{\partial y_0}\frac{\partial x}{\partial x_0} \right)$$

$$= \frac{\partial x}{\partial x_0}\left(\frac{\partial x}{\partial x_0}\frac{\partial y}{\partial y_0}\frac{\partial z}{\partial z_0} + \frac{\partial x}{\partial y_0}\frac{\partial y}{\partial z_0}\frac{\partial z}{\partial x_0} + \frac{\partial x}{\partial z_0}\frac{\partial y}{\partial x_0}\frac{\partial z}{\partial y_0} \right.$$

$$\left. - \frac{\partial x}{\partial x_0}\frac{\partial y}{\partial z_0}\frac{\partial z}{\partial y_0} - \frac{\partial x}{\partial y_0}\frac{\partial y}{\partial x_0}\frac{\partial z}{\partial z_0} - \frac{\partial x}{\partial z_0}\frac{\partial y}{\partial y_0}\frac{\partial z}{\partial x_0} \right) = \frac{\partial x}{\partial x_0}\frac{\partial(x,y,z)}{\partial(x_0,y_0,z_0)}$$

逆行列成分.2

12成分

$$
\begin{aligned}
a_{13}a_{32}-a_{12}a_{33} &= \left(\frac{\partial x}{\partial y_0}\frac{\partial y}{\partial z_0}-\frac{\partial x}{\partial z_0}\frac{\partial y}{\partial y_0}\right)\left(\frac{\partial z}{\partial x_0}\frac{\partial x}{\partial y_0}-\frac{\partial z}{\partial y_0}\frac{\partial x}{\partial x_0}\right) \\
&\quad -\left(\frac{\partial z}{\partial y_0}\frac{\partial x}{\partial z_0}-\frac{\partial z}{\partial z_0}\frac{\partial x}{\partial y_0}\right)\left(\frac{\partial x}{\partial x_0}\frac{\partial y}{\partial y_0}-\frac{\partial x}{\partial y_0}\frac{\partial y}{\partial x_0}\right) \\
&= \frac{\partial x}{\partial y_0}\left(\frac{\partial x}{\partial x_0}\frac{\partial y}{\partial y_0}\frac{\partial z}{\partial z_0}+\frac{\partial x}{\partial y_0}\frac{\partial y}{\partial z_0}\frac{\partial z}{\partial x_0}+\frac{\partial x}{\partial z_0}\frac{\partial y}{\partial x_0}\frac{\partial z}{\partial y_0}\right. \\
&\quad \left.-\frac{\partial x}{\partial x_0}\frac{\partial y}{\partial z_0}\frac{\partial z}{\partial y_0}-\frac{\partial x}{\partial y_0}\frac{\partial y}{\partial x_0}\frac{\partial z}{\partial z_0}-\frac{\partial x}{\partial z_0}\frac{\partial y}{\partial y_0}\frac{\partial z}{\partial x_0}\right)=\frac{\partial x}{\partial y_0}\frac{\partial(x,y,z)}{\partial(x_0,y_0,z_0)}
\end{aligned}
$$

その他の成分も同様な結果である。

逆行列の結果

逆行列は以下のように求まる。

$$\mathbf{A}^{-1}=\frac{1}{\left(\frac{\partial(x,y,z)}{\partial(x_0,y_0,z_0)}\right)^2}\begin{pmatrix}\frac{\partial x}{\partial x_0}\frac{\partial(x,y,z)}{\partial(x_0,y_0,z_0)} & \frac{\partial x}{\partial y_0}\frac{\partial(x,y,z)}{\partial(x_0,y_0,z_0)} & \frac{\partial x}{\partial z_0}\frac{\partial(x,y,z)}{\partial(x_0,y_0,z_0)}\\ \frac{\partial x}{\partial y_0}\frac{\partial(x,y,z)}{\partial(x_0,y_0,z_0)} & \frac{\partial y}{\partial y_0}\frac{\partial(x,y,z)}{\partial(x_0,y_0,z_0)} & \frac{\partial z}{\partial y_0}\frac{\partial(x,y,z)}{\partial(x_0,y_0,z_0)}\\ \frac{\partial z}{\partial x_0}\frac{\partial(x,y,z)}{\partial(x_0,y_0,z_0)} & \frac{\partial z}{\partial y_0}\frac{\partial(x,y,z)}{\partial(x_0,y_0,z_0)} & \frac{\partial z}{\partial z_0}\frac{\partial(x,y,z)}{\partial(x_0,y_0,z_0)}\end{pmatrix}$$

共通因子で括り出す。

ラブランジェの不生不滅の法則

$$\mathbf{A}^{-1} = \frac{1}{\dfrac{\partial(x,y,z)}{\partial(x_0,y_0,z_0)}} \begin{pmatrix} \dfrac{\partial x}{\partial x_0} & \dfrac{\partial x}{\partial y_0} & \dfrac{\partial x}{\partial z_0} \\ \dfrac{\partial x}{\partial y_0} & \dfrac{\partial y}{\partial y_0} & \dfrac{\partial z}{\partial y_0} \\ \dfrac{\partial z}{\partial x_0} & \dfrac{\partial z}{\partial y_0} & \dfrac{\partial z}{\partial z_0} \end{pmatrix}$$

よって渦度は以下の式によって求めることができる。

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \frac{1}{\dfrac{\partial(x,y,z)}{\partial(x_0,y_0,z_0)}} \begin{pmatrix} \dfrac{\partial x}{\partial x_0} & \dfrac{\partial x}{\partial y_0} & \dfrac{\partial x}{\partial z_0} \\ \dfrac{\partial x}{\partial y_0} & \dfrac{\partial y}{\partial y_0} & \dfrac{\partial z}{\partial y_0} \\ \dfrac{\partial z}{\partial x_0} & \dfrac{\partial z}{\partial y_0} & \dfrac{\partial z}{\partial z_0} \end{pmatrix} \begin{pmatrix} \omega_{x0} \\ \omega_{y0} \\ \omega_{z0} \end{pmatrix}$$

ラブランジェの不生不滅の法則

$$\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix}=\frac{1}{\dfrac{\partial(x,y,z)}{\partial(x_0,y_0,z_0)}}\begin{pmatrix}\dfrac{\partial x}{\partial x_0} & \dfrac{\partial x}{\partial y_0} & \dfrac{\partial x}{\partial z_0}\\ \dfrac{\partial x}{\partial y_0} & \dfrac{\partial y}{\partial y_0} & \dfrac{\partial z}{\partial y_0}\\ \dfrac{\partial z}{\partial x_0} & \dfrac{\partial z}{\partial y_0} & \dfrac{\partial z}{\partial z_0}\end{pmatrix}\begin{pmatrix}\omega_{x0}\\ \omega_{y0}\\ \omega_{z0}\end{pmatrix}$$

連続方程式 ⇒ より

$$\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix}=\frac{\rho}{\rho_0}\begin{pmatrix}\dfrac{\partial x}{\partial x_0} & \dfrac{\partial x}{\partial y_0} & \dfrac{\partial x}{\partial z_0}\\ \dfrac{\partial y}{\partial x_0} & \dfrac{\partial y}{\partial y_0} & \dfrac{\partial y}{\partial z_0}\\ \dfrac{\partial z}{\partial x_0} & \dfrac{\partial z}{\partial y_0} & \dfrac{\partial z}{\partial z_0}\end{pmatrix}\begin{pmatrix}\omega_{x0}\\ \omega_{y0}\\ \omega_{z0}\end{pmatrix}$$

導出式の意味

◯過去の任意の時刻t_0において渦度ベクトルがゼロの場合、この結果からそれ以降の如何なる時刻においても常に渦度ベクトルはゼロである。

◯過去の任意の時刻t_0において渦度ベクトルがゼロでない場合、この結果からそれ以降の如何なる時刻においても常に渦度ベクトルはゼロにならない。